LECTURE III

Bi-Hamiltonian chains and it projections

Maciej Błaszak

Poznań University, Poland

Bi-Hamiltonian chains

- Let (M, Π) be a Poisson manifold with Π of constant rank (almost everywhere), with Poisson algebra given by the bracket

$$\{F_1, F_2\}_\Pi = \Pi(dF_1, dF_2) = \langle dF_1, \Pi dF_2 \rangle.$$

- **Remark.** In local Darboux coordinates $(q_1, \dots, q_n, p_1, \dots, p_n, c_1, \dots, c_m)$ Π takes the form

$$\Pi = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where coordinates c_i are Casimir functions.

Definition

A linear combination $\Pi_\lambda = \pi_1 - \lambda \Pi_0$ $(\lambda \in \mathbb{R})$ of two Poisson tensors Π_0, Π_1 is called a Poisson pencil if Π_λ is Poisson for any value of λ. Then Π_0 and Π_1 are called compatible.

Bi-Hamiltonian chains

- Given a Π_λ we can often construct a sequence of vector fields X_i that have two Hamiltonian representations (bi-Hamiltonian chains)

$$X_i = \Pi_1 dH_i = \Pi_0 dH_{i+1},$$

where $H_i \in C^\infty(M)$ are called Hamiltonians of a chain.

- Consider a bi-Poisson manifold (M, Π_0, Π_1) of dim $M = 2n + m$, where Π_0, Π_1 is a pair of compatible Poisson tensors of rank $2n$. We further assume that Π_λ admits m Casimir functions which are polynomial in λ

$$H^{(k)}(\lambda) = \sum_{i=0}^{n_k} H_i^{(k)} \lambda^{n_k - i}, \qquad k = 1, ..., m$$

so, that $n_1 + ... + n_m = n$ and $H_i^{(k)}$ are functionally independent.

- The collection of n bi-Hamiltonian vector fields

$$\Pi_\lambda dH^{(k)}(\lambda)=0 \Longleftrightarrow \begin{array}{ccccc} \Pi_0 dH_0^{(k)} & = & 0 & & \\ \Pi_0 dH_1^{(k)} & = & X_1^{(k)} & = & \Pi_1 dH_0^{(k)} \\ & & \vdots & & \\ \Pi_0 dH_{n_k}^{(k)} & = & X_{n_k}^{(k)} & = & \Pi_1 dH_{n_k-1}^{(k)} \\ & & 0 & = & \Pi_1 dH_{n_k}^{(k)} \end{array} \qquad (3.1)$$

where $k=1,\ldots,m$, define bi-Hamiltonian systems of Gel'fand-Zakharevich type. Notice that each chain starts from a Casimir of Π_0 and terminates with a Casimir of Π_1.

Lemma

All $H_i^{(k)}$ pairwise commute wit respect to Π_0 and Π_1.

Bi-Hamiltonian chains

GZ bi-Hamiltonian chain $\Longleftrightarrow$ Liouville integrable system

- Having *GZ* bi-Hamiltonian system the crucial toward construction of separation coordinates is the projection of second Poisson tensor Π_1 onto the symplectic foliation of the first Poisson tensor Π_0.

Poisson projections onto submanifolds

- Consider manifold M of $\dim M = m$ and foliation S consisting of leaves S_ν parametrized by $\nu \in \mathbb{R}^r$, so r is codimension of every leave.
- Let $\mathcal{Z}$ be a distribution transversal to S, i.e.

$$T_x M = T_x S_\nu \oplus \mathcal{Z}_x$$

  where S_ν is a leave that passes through x.
- It defines a decomposition

$$TM \ni X = X_\| + X_\perp, \qquad \left(X_\|\right)_x \in T_x S_\nu, \quad \left(X_\perp\right)_x \in \mathcal{Z}_x$$

- and induces splitting of dual space

$$T_x^* M = T_x^* S_\nu \oplus \mathcal{Z}_x^*,$$

  where $T_x^* S_\nu$ annihilates $\mathcal{Z}_x$ and $\mathcal{Z}_x^*$ annihilates $T_x S_\nu$.

Poisson projections onto submanifolds

- Thus, any one-form α on M has a unique decomposition

$$T^*M \ni \alpha = \alpha_{\parallel} + \alpha_{\perp}, \qquad (\alpha_{\parallel})_x \in T^*_x S_\nu, \quad (\alpha_{\perp})_x \in \mathcal{Z}^*_x.$$

Definition

A function $F \in C(M)$ is called $\mathcal{Z}$-invariant if $L_Z F = Z(F) = 0$ for any $Z \in \mathcal{Z}$. Obviously $dF \in T^*S$.

Definition

The Poisson tensor Π is said to be $\mathcal{Z}$-invariant if

$$L_Z\{F, G\}_\Pi = 0$$

for any pair of $\mathcal{Z}$-invariant functions F, G and any $Z \in \mathcal{Z}$.

Poisson projections onto submanifolds

- For any Poisson tensor Π on M define the following bivector Π_D

$$\Pi_D(\alpha,\beta) := \Pi(\alpha_{\|},\beta_{\|}), \qquad \alpha,\beta \in T^*M \tag{3.2}$$

Π_D - deformation of Π.

Lemma

The image of Π_D is tangent to the foliation S.

- Indeed,

$$\langle \alpha, \Pi_D\beta\rangle = \langle \alpha_{\|}, \Pi\beta_{\|}\rangle = 0 \;\text{ for }\; \beta \in \mathcal{Z}^* \implies \mathcal{Z}^* \subset \ker \Pi_D.$$

- So, Π_D can be naturally restricted to any leave S_ν : $\pi = \Pi_{D|S}$.

Theorem

If Π is $\mathcal{Z}$-invariant, then Π_D is Poisson and so π.

Proof.

From $\mathcal{Z}$-invariant of Π : $L_Z\langle dF_{\|}, \Pi dG_{\|}\rangle = 0$, so

$$d\ \langle dF_{\|}, \Pi dG_{\|}\rangle \in T^*S \Longrightarrow \left(d\ \langle dF_{\|}, \Pi dG_{\|}\rangle\right)_{\|} = d\ \langle dF_{\|}, \Pi dG_{\|}\rangle,$$

hence, the Jacobi identity

$$\{\{F, G\}_{\Pi_D} + c.p. = \langle\left(d\ \langle dF_{\|}, \Pi dG_{\|}\rangle\right)_{\|}, \Pi dH_{\|}\rangle + c.p. = \langle d\ \langle dF_{\|}, \Pi dG_{\|}\rangle, \Gamma$$

is fulfilled. □

- **Observation.** Annihilator $\mathcal{Z}^*$ of TS is defined as soon as S is determined.

Definition

Distribution $D = \Pi(\mathcal{Z}^*)$, associated with the foliation S, is called Dirac distribution.

Poisson projections onto submanifolds

- Two limited cases are possible:
  1. $TM = D \oplus TS$ Dirac case
  2. $D \subset TS$ tangent case.
- Let S be parametrized by $\varphi_i(x) \in C(M)$:

$$S_\nu = \{x \in M : \varphi_i(x) = \nu_i, \quad i = 1, ..., r\},$$

  so $\{d\varphi_i\} -$ basis in $\mathcal{Z}^*$.
- Denote by $\{Z_i\}$ a basis dual to $\{d\varphi_i\}$, so $Z_i(\varphi_j) = \delta_{ij}$. Then,

$$X_\parallel = X - \sum_{i=1}^{r} X(\varphi_i)Z_i, \quad \alpha_\parallel = \alpha - \sum_{i=1}^{r} \alpha(Z_i)d\varphi_i.$$

- Obviously $X_\parallel(\varphi_i) = 0$ and $\alpha_\parallel(Z_i) = 0$.

Poisson projections onto submanifolds

- Then, from $\Pi_D(\alpha,\beta) = \Pi(\alpha_{\|},\beta_{\|})$ we get

$$\Pi_D = \Pi - \sum_{i=1}^{r} X_i \wedge Z_i + \frac{1}{2}\sum_{i,j=1}^{r} \varphi_{ij} Z_i \wedge Z_j \tag{3.3}$$

  where $X_i = \Pi d\varphi_i$, $\varphi_{ij} = \{\varphi_i, \varphi_j\}_\Pi$.
- In the Dirac case we have a canonical choice of $\mathcal{Z} = D$, as all X_i are transversal to S and are linearly independent ($\det(\varphi_{ij}) \neq 0$). Moreover Π is naturally $\mathcal{Z}$-invariant since $L_{X_i}\Pi = 0$.
- $\varphi_i(x)$ are then "second class constraints" in Dirac terminology.
- The basis $\{Z_i\}$ dual to $\{d\varphi_i\}$ can be expressed by X_i:

$$Z_i = \sum_{j=1}^{r} \left(\varphi^{-1}\right)_{ji} X_j, \qquad Z_j(\varphi_i) = \delta_{ij}.$$

- Π_D (3.3) attains the form

$$\Pi_D = \Pi - \frac{1}{2}\sum_{i,j=1}^{r} \left(\varphi^{-1}\right)_{ij} X_j \wedge X_i.$$

- This Poisson tensor defines the following bracket on $C(M)$:

$$\{F, G\}_{\Pi_D} = \{F, G\}_{\Pi} - \sum_{i,j=1}^{r} \{F, \varphi_i\}_{\Pi} \left(\varphi^{-1}\right)_{ij} \{\varphi_j, G\}_{\Pi}$$

known as a Dirac bracket.

- In tangent case all X_i are tangent to S, i.e. $X_i(\varphi_j) = \Pi(d\varphi_i, d\varphi_j) = 0$, so

$$\Pi_D = \Pi - \sum_{i=1}^{r} X_i \wedge Z_i$$

and Z_i must be find separately from case to case.

From bi-Hamiltonian to quasi-bi-Hamiltonian chains

- Our foliation is symplectic foliation of Π_0, so $\mathcal{Z}^* = Ker\ \Pi_0 = Sp\{dc_i\}$.
- For GZ-bi-Hamiltonian chains $\Pi_1(dc_i, dc_j) = 0$, so we are in tangent case when project Π_1 onto S.
- Assume we found $\mathcal{Z}$ transversal to S and such that Π_1 is $\mathcal{Z}$ - invariant. Then Π_D is Poisson with $Ker\ \Pi_0 \subseteq Ker\ \Pi_D$, so both tensors can be restricted to S.
- Let

$$\pi_0 = \Pi_{0|S}, \quad \pi_1 = \Pi_{D|S}.$$

From bi-Hamiltonian to quasi-bi-Hamiltonian chains

Theorem

Bi-Hamiltonian chains (3.1) restricted to S take the form of quasi-bi-Hamiltonian chains

$$\pi_1 dh_i^{(k)} = \pi_0 \left(dh_{i+1}^{(k)} - \sum_{j=1}^{m} \alpha_{ij}^{(k)} dh_1^{(j)} \right), \tag{3.4}$$

where

$$\alpha_{ij}^{(k)} = \left(Z_j(H_i^{(k)}) \right)_{|S}, \quad h_i^{(k)} = (H_i^{(k)})_{|S}.$$

- It follows from the fact that $\Pi_1 = \Pi_D + \sum_{k=1}^{m} X_1^{(k)} \wedge Z_k$.

From bi-Hamiltonian to quasi-bi-Hamiltonian chains

Lemma

$$\{H_i^{(k)}, H_j^{(r)}\}_{\Pi_D} = 0.$$

Theorem

Necessary and sufficient condition for compatibility of Π_0 *and* Π_D *(hence* π_0 *and* π_1*) is*

$$L_{Z_i}\Pi_0 = 0.$$

- Z_i are symmetries of Π_0 (so Π_0 is $\mathcal{Z}$ - invariant), hence $\mathcal{Z}$ is integrable distribution and $[Z_i, Z_j] = 0$.

N-manifolds

Definition

$\omega N-$ manofold is a bi-Poisson manifold (π_0, π_1) with two compatible Poisson tensors, where at least one (say π_0) is nondegenerated.

- So M is endowed with a symplectic form $\omega = \omega_0 = \pi^{-1}$ and the $(1,1)-$ tensor field $N = \pi_1 \omega_0$.
- From compatibility of π_0 and π_1 follows that Nijenhuis torsion of N vanishes

$$T(N)(X,Y) = [NX,NY] - N[NX,y] - N[X,NY] + N^2[X,Y] = 0$$

$$\Updownarrow$$

$$L_{NX}N = NL_X N$$

and hence, the second two-form

$$\omega_1 = \omega_0 \pi_1 \omega_0$$

is closed.

N-manifolds

- We further restrict to generic case, when N has at every point n distinct eigenvalues which are functionally independent.
- It means that π_1 is also nondegenerated and ω_1 is symplectic.
- Notice that

$$\pi_1 = N\pi_0, \quad \omega_1 = N^*\omega_0, \quad N^* = \omega_0\pi_1.$$

- Moreover,

$$\{f, g\}_{\pi_i} = \omega_i(X_f, X_g), \qquad X_h = \pi_0 dh, \quad i = 1, 2.$$

- Let us come back to our quasi-bi-Hamiltonian chain (3.4). The distribution tangent to the foliation defined by $(h_1^{(1)}, \dots, h_{n_m}^{(m)})$ is bi-Lagrangian:

$$\omega_0(X_{h_i^{(k)}}, X_{h_j^{(r)}}) = \omega_1(X_{h_i^{(k)}}, X_{h_j^{(r)}}) = 0.$$

- Moreover, quasi-bi-Hamiltonian equations can be put into one matrix equation

$$N^* dh_i = \sum_{j=1}^{m} F_{ij} dh_j \Longleftrightarrow N^* dh = F dh$$

where $(h_1, ..., h_n) = (h_1^{(1)}, ..., h_{n_m}^{(m)})$ and $F-$ control matrix (recursion matrix).